Table 1 Hall voltages when $\sigma = \sigma_0 + C|j|$, $H_a = 3$

$\omega\tau$	Cu_0B_z	$E_x/\langle u\rangle B_z$	$-\omega\tau$
1	0.5	−1.03	−1.00
10	0.5	−10.02	−10.00
10	0.9	−10.05	−10.00

Analysis

In order to couple the nonequilibrium effect to the flow field in a way simple enough to permit solutions to the flow problem, a one-fluid system of equations has been evolved in which the nonequilibrium effect is accounted for by allowing the electrical conductivity to be a function of the current density.[6]

Two assumptions have been made as to the functional dependency. They are

$$\sigma = \sigma_0 + C|j| \tag{2}$$

$$\sigma = \sigma_R|j|^n \qquad n \geq 0 \tag{3}$$

where the first would seem appropriate when σ is close to σ_0, and the second where σ is considerably changed from its thermal value. Before proceeding to the question at hand it may be of interest to note that, when the foregoing assumptions are combined with the generalized Ohm's law, one finds limiting values that the constants C and n may assume. They are

$$C < 1/|E_y - uB_0| \qquad n < 1 \tag{4}$$

These restrictions are not related to the flow problem but only to the Ohm's law. Further restriction has been found for C by attempting to solve the flow problem for values of C close to but not equal to its value given in Eq. (4).[6] It has been found that no combination of the parameters involved will permit a solution.

Restricting C and n to values below their critical values, the Hartmann flow problem has been considered where it is assumed that nonequilibrium effects exist, $\sigma = \sigma(j)$, and the Hall effect also is present, $\omega\tau \neq 0$. For each case, one obtains a pair of simultaneous, nonlinear, ordinary differential equations with three arbitrary parameters. These parameters, respectively, the axial pressure gradient, the transverse pressure gradient, and the Hall field, are determined iteratively by requiring the average transverse flow to be zero, the average axial current flow to be zero, and the flow velocity at the channel walls to be zero. Solution of these equations, along with the required iterations, has been obtained on an analog computer.

Results

In general, the Hall electric field will be largest when the Faraday electric field L_y is zero. Thus, we will only consider cases for which this assumption has been made. The first set of calculations correspond to the $\sigma(j)$ assumption of Eq. (2), and are presented in Table 1. Additional calculations have been carried out for the $\sigma(j)$ assumption of Eq. (3). In this instance, only the $n = \frac{1}{2}$ case has been evaluated and is presented in Table 2. We recall from Eq. (1) that $E_x/\langle u\rangle B_z = -\omega\tau$ when all nonequilibrium effects are neglected, so that the results as calculated from the present model[6] are compared in the tables to what one would expect from the simple theory.

Table 2 Hall voltages when $\sigma = \sigma_R|j|^{1/2}$, $(\sigma_R/\sigma_0)H_a = 3$

$\omega\tau$	$E_x/\langle u\rangle B_z$	$-\omega\tau$
1	−1.06	−1.00
10	−10.09	−10.00

Discussion

Reviewing our calculated results, we observe that the Hall voltages derived are *larger* than what would have been expected. Also, the larger the Hall effect $\omega\tau$, the smaller this increase, and the closer Cu_0B_z comes to its limiting value the larger the increase becomes. In addition, we note that when the nonequilibrium effect is stronger, $\langle u\rangle$ is larger than when it has been neglected,[6] so that the *dimensional* Hall electric field will be even greater than shown in the tables.

As was noted earlier, when conductivity nonuniformities exist normal to the generator electrodes, one can show that the Hall potential should be reduced either when infinitely finely segmented electrodes are assumed[2] or when finite electrodes are accounted for.[3] By contrast, however, we have shown here that, when such nonuniformities exist normal to the insulators, the Hall voltage is *increased* in spite of the attendant axial current flows. Therefore, one might conclude that the severe reductions predicted by Kerrebrock[3] may be somewhat relieved by the results obtained in the present study.

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A Simple Universal Velocity Profile Equation

Ralph A. Burton*

Southwest Research Institute, San Antonio, Texas

NUMEROUS authors have attempted to fill the need for a continuous universal velocity profile expression, reaching all the way from the wall out into the turbulent flow. These include Miles,[1] van Driest,[2] Szablewski,[3] Reichardt,[4] and Ng.[5] Aside from the fact that none of these precisely represent the others, all are relatively difficult functionally and make hand calculation virtually impossible when one tries to use them to compute shear stress from velocity data. To provide a simpler expression for such work, we have used the expression

$$Y^+ = U^+ + (U^+/a)^n = U^+ + (U^+/8.74)^7 \tag{1}$$

Here $U^+ \equiv u(\rho/\tau)^{1/2}$ and $Y^+ \equiv (y/\nu)(\tau/\rho)^{1/2}$, where u is

Received November 30, 1964; revision received December 14, 1964. This work was done under Contract No. AT(11-1)-1228, under Atomic Energy Commission sponsorship, with Nicholas Grossman as project monitor. Herbert J. Carper of the Institute has provided much appreciated help with the graphs and calculations.

* Staff Scientist and Manager, Lubrication Research Section.

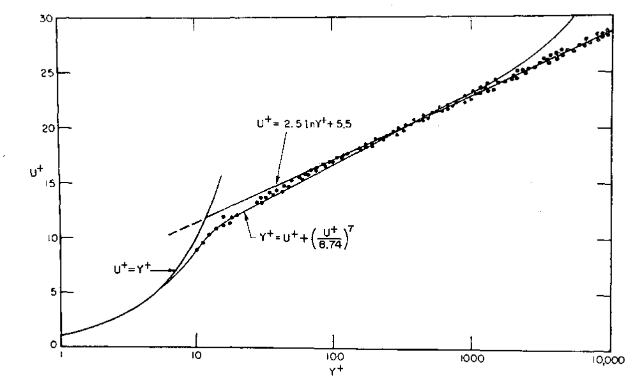

Fig. 1 Universal velocity profile showing logarithmic approximation and proposed approximation as compared with Nikuradse's data.

velocity parallel to the wall, y is the coordinate normal to the wall, ρ is fluid density, τ is wall shear stress, and ν is kinematic viscosity. Somewhat unexpectedly, this relationship was found to fit available experimental data well. As shown in Fig. 1, it follows the data of Nikuradse[6] up to a value of $Y^+ = 500$ and possibly as high as $Y^+ = 1000$. To obtain a better assessment of its effectiveness in the transition region near $Y^+ = 10$, this has been expanded as shown in Fig. 2, and the data of Reichardt[4] and Laufer (as presented in Ref. 1) have also been included to show the range of disagreement in experimental values. There is exceptionally good agreement between the selected function and Laufer's data, and it would be very satisfying if we could convince ourselves that these recent measurements are the most accurate.

Defining $C_f \equiv 2\tau/\rho u^2$, and $R \equiv uy/\nu$, Eq. (1) leads to the following friction law:

$$R = (2/C_f) + [(2/C_f)^4/(8.74)^7] \tag{2}$$

A defining equation for eddy viscosity η may be written as follows:

$$\rho(\nu + \eta)(\partial u/\partial y) = \tau \tag{3}$$

The shear stress τ may be assumed constant, since this is one of the essential conditions of the universal velocity profile. Combining Eqs. (1) and (2),

$$\eta/\nu = 0.801\ (U^+/8.74)^6 \tag{4}$$

Mixing length l may be defined by

$$\eta \equiv l^2|\partial u/\partial y| \tag{5}$$

Combining this with the preceding relationship for eddy viscosity [Eq. (4)], and defining $\tau l^2/\rho\nu^2 \equiv (L^+)^2$

$$(L^+)^2 = 0.801(U^+/8.74)^6[1 + 0.801(U^+/8.74)^6] \tag{6}$$

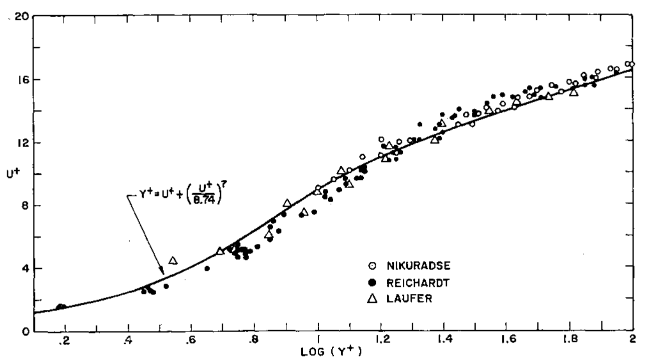

Fig. 2 Universal velocity profile expanded to show transition zone.

It is not convenient to solve this explicitly for L^+ in terms of Y^+, but for the special cases of very large and very small Y^+ these become, respectively, for large Y^+,

$$L^+ = 0.801(U^+/8.74)^6 = 0.801(Y^+)^{6/7} \tag{7}$$

and for very small Y^+

$$L^+ = (0.801)^{1/2}(U^+/8.74)^3 = (0.801)^{1/2}(Y^+/8.74)^3 \tag{8}$$

Note that for the very small Y^+ this follows a cubic relation for which Emmons[7] has given some substantiation in terms of reported measurements of eddy dimension. At the higher values of Y^+ the relationship is very nearly linear, as is ordinarily assumed in mixing length theory. The relationship between L^+ and Y^+ is best illustrated in Fig. 3, where the predictions of Eq. (6) combined with Eq. (1) are shown merging into the cubic relationship at the small Y^+ and approaching (very nearly paralleling) the ordinarily assumed mixing length relationship of $L^+ = 0.4Y^+$. On the basis of these observations we suggest that the simple profile represented by Eq. (1) not only provides a good direct fit of empirical velocity profile data, but also appears to make sense when viewed in terms of the mixing length relationship that it carries with it implicitly.

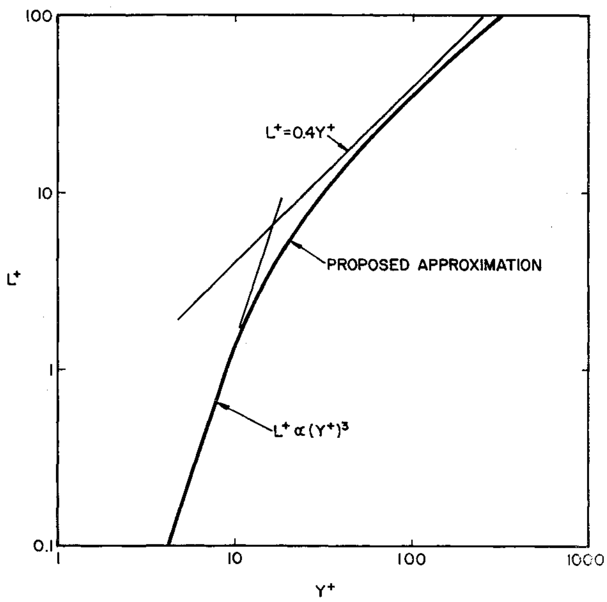

Fig. 3 Plot of dimensionless mixing length.

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